รายงานการวิจัย

ภาวะคู่กันของปริภูมิเบิร์กแมนนัยทั่วไป

The Duality of a Generalized Bergman Space

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คณะศิลปศาสตร์

มหาวิทยาลัยเทคโนโลยีราชมงคลศรีวิชัย

ได้รับการสนับสนุนทุนวิจัยจากมหาวิทยาลัยเทคโนโลยีราชมงคลศรีวิชัย

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ภาวะคู่กันของปริภูมิเบิร์กแมนนัยทั่วไป

มาริสา เส็นเหมาะ[1]

บทคัดย่อ

ปริภูมิเบิร์กแมนคือปริภูมิของฟังก์ชันโฮโลมอร์ฟิกซึ่งกำลังสองสามารถหาปริพันธ์ได้เมื่อเทียบกับเมเชอร์ dv_α โดยที่ $dv_\alpha = c_\alpha(1-|z|^2)^\alpha$ นั่นคือ

$$\mathrm{H}L^2(\mathrm{B}, dv_\alpha) = \{ f \mid f \in L^2(\mathrm{B}, dv_\alpha) \cap \mathrm{H}(\mathrm{B}) \}$$

ปริภูมิเบิร์กแมนจะไม่เป็นปริภูมิศูนย์ก็ต่อเมื่อ $\alpha > -1$ อย่างไรก็ตามจากการพิจารณาค่าของรีโปรดักซิ่งเคอเนล $K(w,z) = \dfrac{1}{\pi(1-\langle z,w\rangle)^{\alpha+2}}$ ทำให้ทราบว่า $K(w,z)$ ยังคงนิยามอย่างบวกได้จนถึงกรณี $-2 < \alpha \le -1$ และได้นิยามปริภูมิเบิร์กแมนเชิงทั่วไปไว้ดังนี้

$$HL^2(\mathrm{B}, \alpha) = \left\{ f \in HL^2\left(\mathrm{B}, dv_{\alpha+2}\right) : z\frac{df}{dz} \in HL^2\left(\mathrm{B}, dv_{\alpha+2}\right) \right\}$$

จาก [Chailuek,K and Hall,B] ผู้เขียนได้ศึกษาเกี่ยวกับสมบัติบางประการของปริภูมิเบิร์กแมนเชิงนัยทั่วไปซึ่งรวมถึงการศึกษาภาวะคู่กันของปริภูมิเบิร์กแมนเชิงนัยทั่วไป ในกรณีที่ $\alpha, \beta > -2$ ไว้แล้ว

ในการศึกษาครั้งนี้ เราได้ศึกษาเพิ่มเติมถึงภาวะคู่กันของปริภูมิเบิร์กแมนนัยทั่วไป นั่นคือจะแสดงให้เห็นว่าปริภูมิเบิร์กแมน ในกรณีที่ α, β มีค่าใดๆแล้ว ก็ยังคงมีสมบัติการเป็นภาวะคู่กัน

คำสำคัญ: ภาวะคู่กัน

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Marisa Senmoh

THE DUALITY OF A GENERALIZED BERGMAN SPACE

Marisa Senmoh[1]

Abstract

A Bergman space $\mathcal{H}L^2(\mathbb{B},\ dv_\alpha)$ is the space consisting of all holomorphic functions on the unit ball $\mathbb{B}$ which are square- integrable with respect to dv_α where $dv_\alpha = c_\alpha(1-|z|^2)^\alpha$. The space is non-zero when $\alpha > -1$. However, these spaces can be extended to the case $-2 < \alpha \leq -1$ by defining a generalized Bergman space

$$HL^2(\mathbb{B},\ \alpha) = \left\{ f \in \mathcal{H}L^2(\mathbb{B},\ dv_{\alpha+2}) : z\frac{df}{dz} \in \mathcal{H}L^2(\mathbb{B},\ dv_{\alpha+2}) \right\}$$

which $HL^2(\mathbb{B},\ \alpha) = \mathcal{H}L^2(\mathbb{B},\ dv_\alpha)$ when $\alpha > -1$ and $HL^2(\mathbb{B},\ \alpha)$ is non-zero when $-2 < \alpha \leq -1$.By [Chailuek,K and Hall, B], the authers proved some properties of a generalized Bergman space and including the duality of a generalized Bergman space for $\alpha, \beta > -2$

In this reserch, we are interested in the duality of a generalized Bergman space for all α, β.

Keyword: Duality

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CONTENT

CHAPTER 1

Introduction

Let $\mathbb{B}^d = \left\{ z = (z_1,\, z_2,\, \ldots\, z_d) \in \mathbb{C}^d \colon \|z\| = \sqrt[d]{\sum_{i=1}^{d} |z_i|^2} < 1 \right\}$ be the open unit ball in $\mathbb{C}^n$. We define the measure

$$d\mu_\lambda = c_\lambda (1 - |z|^2)^{\lambda - (d+1)}\, dz$$

where c_λ is the normalization factor defined by $c_\lambda = \dfrac{\Gamma(\lambda)}{\pi^d \Gamma(\lambda - d)}$, $\lambda > d$. Denote by $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$, the weighted Bergman space consisting of all holomorphic functions on $\mathbb{B}^d$ that are square-integrable with respect to the measure μ_λ. These spaces are Hilbert spaces.

The condition $\lambda > d$ is due to the fact that the measure μ_λ is finite if and only if $\lambda > d$. When the measure is finite, all bounded holomorphic functions are square-integrable and, more importantly, the constant c_λ makes the measure is a probability measure. However, when the measure is infinite, there are no nonzero holomorphic functions that are square-integrable with respect to μ_λ.

For $\lambda > d$ and by the Riesz representation, any function $f \in \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ can be represented as

$$f(z) = \int_{\mathbb{B}^d} K_\lambda(z, w) f(w)\, d\mu_\lambda(w)$$

where $K_\lambda(z, w) = \frac{1}{(1 - z\cdot\overline{w})^\lambda}$ is called the reproducing kernel for this space.

Consider the formula for the reproducing kernel $K(w, z) = \frac{1}{(1 - z\cdot\overline{w})^\lambda}$. It is positive definite for all $\lambda > 0$, not only $\lambda > d$. This is an evidence to support that the space $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ can be extended to $\lambda > 0$ as "reproducing kernel Hilbert spaces" .

According to Theorem 4 in [Chailuek,K and Hall,B], we can define a holomorphic Sobolev space (or Besov space) as follows. Let $n = \left\lceil \dfrac{d}{2} \right\rceil$, for all $\lambda > 0$, define

$$H(\mathbb{B}^d, \lambda) = \{ f \colon \mathbb{B}^d \to \mathbb{C} \,|\, N^k f \in \mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n}),\, 0 \le k \le n \}$$

where N denote the number operator

$$N = \sum_{i=1}^{d} z_i \frac{\partial}{\partial z_i}.$$

Then $\langle f, g \rangle_\lambda = \langle Af, Bg \rangle_{\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})}$ where

$$\begin{aligned} A &= \left(I + \frac{N}{\lambda + n}\right)\left(I + \frac{N}{\lambda + n + 1}\right)\cdots\left(I + \frac{N}{\lambda + 2n - 1}\right) \\ B &= \left(I + \frac{N}{\lambda}\right)\left(I + \frac{N}{\lambda + 1}\right)\cdots\left(I + \frac{N}{\lambda + n - 1}\right) \end{aligned}$$

defines an inner product on $H(\mathbb{B}^d, \lambda)$ and , with respect to this inner product, $H(\mathbb{B}^d, \lambda)$ is a complete space whose reproducing kernel is also given by $K_\lambda(z, w) = \dfrac{1}{(1 - z \cdot \overline{w})^\lambda}$. Moreover, $H(\mathbb{B}^d, \lambda)$ is identical to $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ when $\lambda > d$.

By the definition of a generalized Bergman space. In this research, we will show that the duality of a generalized Bergman space can be proved by direct computation and boundedness of coefficients.

CHAPTER 2
Preliminaries

In this chapter, we first collect some basic knowledge and the notations of operators used in this research.

Definition 1. Let X be a vector space over a field $\mathbb{F}$. A function $\|\cdot\| : X \mapsto [0,\infty)$ is said to be a **norm** on X if

(i) $\|x\| = 0$ if and only if $x = 0$

(ii) $\|cx\| = |c|\|x\|$ for any $x \in X$ and $c \in \mathbb{F}$

(iii) $\|x+y\| \leq \|x\| + \|y\|$ for any $x, y \in X$.

A vector space equipped with a norm is called a **normed linear space**, or simply a **normed space**. Property (iii) is referred to as the *triangle inequality*.

Definition 2. The metric space (X, d) is said to be **complete** if every Cauchy sequence in X converges (that is has a limit which is an element of X). That is if $d(x_n, x_m) \to 0$ as $m, n \to \infty$ then $\{x_n\}$ must converge also in X.

Definition 3. A **Banach Space** is a normed linear space which is complete in the metric defined by its norm. That is $d(x, y) = \|x - y\|$.

Definition 4. An **inner product** on a vector space V is a function that associates a complex number $\langle u, v\rangle$ with each pair of vector u and v in V in such a way that the following axioms are satisfied for all vectors u, v and w in V and all scalars k.

(i) $\langle u, v\rangle = \langle v, u\rangle$

(ii) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

(iii) $\langle ku, v\rangle = k\langle u, v\rangle$

(iv) $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ if and only if $v = 0$.

A vector space equipped with an inner product is called an **inner product space**.
So if we define $\|v\| = \sqrt{\langle v, v\rangle}$ then $\|\cdot\|$ is a norm on V.

Definition 5. For $1 \geq p < \infty$,the $\mathcal{L}^{\mathcal{P}}(X, \mu)$-space is the collection of all functions $f : X \to \mathbb{C}$ such that

$$\int_X \|f(z)\|^p d\mu(z) < \infty.$$

We define $L^p(X, \mu)$ to be the space of all equivalence classes of functions in $L^p(X, \mu)$ under the relation fg if and only if $f = g$ almost everywhere with respect to the measure μ

Definition 6. A **Hilbert space** is an inner product space which is complete with respect to the norm given by the inner product.

Theorem 1. **(Riesz Representation)** *If L is a bounded linear functional on a Hilbert space H, then there exists a unique $y \in H$ such that*

$$L(x) = \langle x, y \rangle \qquad \textit{for each } x \in H$$

Moreover $\|L\| = \|y\|$.

Theorem 2. **(Hölder inequality)** *If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq (\sum_{i=1}^{n} |a_i|^p)^{\frac{1}{p}} (\sum_{i=1}^{n} |b_i|^q)^{\frac{1}{q}}$$

Definition 7. Let X be a norm linear space. Denote by X^* the set of all bounded linear functional on X. We call X^* the **dual space** of X

Theorem 3. **(Duality of Bergman spaces)** *A Bergman space can be represented by the dual of another Bergman space by the following theorem. (See Zhu,K Theorem 2.12) For $\alpha, \beta > d$,*

$$\mathcal{H}L^2(\mathbb{B}^d, \mu_\alpha)^* = \mathcal{H}L^2(\mathbb{B}^d, \mu_\beta)$$

under the inner product

$$\langle f, g \rangle_{\mathcal{H}L^2(\mathbb{B}^d, \mu_\gamma)} = \int_{\mathbb{B}^d} f(z)\overline{g(z)}\, d\mu_\gamma(z),$$

for $f \in \mathcal{H}L^2(\mathbb{B}^d, \mu_\alpha)$, $g \in \mathcal{H}L^2(\mathbb{B}^d, \mu_\beta)$ and $\gamma = \dfrac{\alpha + \beta}{2}$.

Duality of generalized Bergman spaces. It should be noted that a Bergman space $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ is a closed subspace of the space $L^2(\mathbb{B}^d, \mu_\lambda)$. However, by its definition, $H(\mathbb{B}^d, \lambda)$ is not defined as a subspace of any L^2 space. Therefore the proof of the duality of Bergman spaces cannot be adopted to $H(\mathbb{B}^d, \lambda)$. However, the duality of generalized Bergman space can be proved by direct computation and boundedness of coefficients.

CHAPTER 3

Main Results

Theorem 4. *For* $\alpha, \beta > 0$

$$H(\mathbb{B}^d, \alpha)^* = H(\mathbb{B}^d, \beta)$$

under the inner product

$$\langle f, g\rangle_\gamma = \int_{\mathbb{B}^d} Af(z)\overline{Bg(z)}\, d\mu_{\gamma+2n}(z),$$

for $f \in H(\mathbb{B}^d, \alpha),\ g \in H(\mathbb{B}^d, \beta)$ *and* $\gamma = \dfrac{\alpha+\beta}{2}$.

Proof. For each $g \in H(\mathbb{B}^d, \beta)$, we define $T_g \colon H(\mathbb{B}^d, \alpha) \to \mathbb{C}$ by

$$T_g(f) = \langle f, g\rangle_\gamma.$$

Next, we will prove that $T_g \in H(\mathbb{B}^d, \alpha)^*$. Consider

$$\begin{aligned}
|T_g(f)| &= |\langle f, g\rangle_\gamma| \\
&= |\langle Af, Bg\rangle_{\mathcal{H}L^2(\mathbb{B}^d, \mu_{\gamma+2n})}| \\
&= c_{\gamma+2n}\left|\int_{\mathbb{B}^d} Af(z)\overline{Bg(z)}(1-|z|^2)^{\gamma+2n}\,(1-|z|^2)^{-(d+1)}dz\right| \\
&\leq c_{\gamma+2n}\int_{\mathbb{B}^d}(1-|z|^2)^{\frac{\alpha+2n}{2}}|Af(z)|(1-|z|^2)^{\frac{\beta+2n}{2}}\overline{|Bg(z)|}\,(1-|z|^2)^{-(d+1)}dz.
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
|T_g(f)| &\leq c_{\gamma+2n}\left(\int_{\mathbb{B}^d}((1-|z|^2)^{\frac{\alpha+2n}{2}}|Af(z)|)^2\,(1-|z|^2)^{-(d+1)}dz\right)^{\frac{1}{2}} \\
&\quad\cdot\left(\int_{\mathbb{B}^d}((1-|z|^2)^{\frac{\beta+2n}{2}}\overline{|Bg(z)|})^2\,(1-|z|^2)^{-(d+1)}dz\right)^{\frac{1}{2}} \\
&= c_{\gamma+2n}\left(\int_{\mathbb{B}^d}|Af(z)|^2(1-|z|^2)^{\alpha+2n}(1-|z|^2)^{-(d+1)}dz\right)^{\frac{1}{2}} \\
&\quad\cdot\left(\int_{\mathbb{B}^d}\overline{|Bg(z)|}^2(1-|z|^2)^{\beta+2n}(1-|z|^2)^{-(d+1)}dz\right)^{\frac{1}{2}} \\
&= c_{\gamma+2n}\|Af(z)\|_{\mathcal{H}L^2(\mathbb{B}^d, \mu_{\alpha+2n})}\|Bg(z)\|_{\mathcal{H}L^2(\mathbb{B}^d, \mu_{\beta+2n})} \\
&= c_{\gamma+2n}\langle Af(z), Af(z)\rangle_{\alpha+2n}\langle Bg(z), Bg(z)\rangle_{\beta+2n}.
\end{aligned}$$

By considering the coefficients in the operators A and B, there exist constants $C_A(n,\alpha)$ and $C_B(n,\beta)$ such that $\langle Af(z), Af(z)\rangle_{\alpha+2n} \leq C_A(n,\alpha)\langle f(z), f(z)\rangle_{\alpha+2n}$ and $\langle Bg(z), Bg(z)\rangle_{\beta+2n} \leq C_B(n,\alpha)\langle g(z), g(z)\rangle_{\beta+2n}$.

Therefore, $|T_g(f)| \leq C\|g\|_{\beta+2n}\|f\|_{\alpha+2n}$ where the constant C is independent of f.

Conversely, let $F \in H(\mathbb{B}^d,\alpha)^*$. By Riesz representation, there exists a function $h \in H(\mathbb{B}^d,\alpha)$ such that $F(f) = \langle f, h\rangle_\alpha$ for all $f \in H(\mathbb{B}^d,\alpha)$. To prove $H(\mathbb{B}^d,\alpha)^* = H(\mathbb{B}^d,\beta)$, we need a function $g \in H(\mathbb{B}^d,\beta)$, instead of $h \in H(\mathbb{B}^d,\alpha)$, such that $F(f) = \langle f, g\rangle_\gamma$. However, by manipulating the coefficients, we obtain that function g.

Consider, for $f \in H(\mathbb{B}^d,\alpha)$,

$$F(f) = \langle f, h\rangle_\alpha = \langle Af, Bh\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\alpha+2n})}.$$

Now the operator A and B can be distributed as

$$A = \sum_{k=1}^{n} R_k N^k + I \text{ and } B = \sum_{k=1}^{n} S_k N^k + I.$$

Therefore,

$$\begin{aligned}
F(f) &= \langle Af, Bh\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\alpha+2n})} \\
&= \left\langle \sum_{k=1}^{n} R_k N^k f + f, \sum_{k=1}^{n} S_k N^k h + h \right\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\alpha+2n})} \\
&= \left\langle \sum_{k=1}^{n} R_k N^k f, \sum_{k=1}^{n} S_k N^k h \right\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\alpha+2n})} + \left\langle \sum_{k=1}^{n} R_k N^k f, h \right\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\alpha+2n})} \\
&\quad + \left\langle f, \sum_{k=1}^{n} S_k N^k h \right\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\alpha+2n})} + \langle f, h\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\alpha+2n})} \\
&= \left\langle \sum_{k=1}^{n} \mathcal{R}_k N^k f, \sum_{k=1}^{n} \mathcal{S}_k N^k Mh \right\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\gamma+2n})} + \left\langle \sum_{k=1}^{n} \mathcal{R}_k N^k f, Mh \right\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\gamma+2n})} \\
&\quad + \left\langle f, \sum_{k=1}^{n} \mathcal{S}_k N^k Mh \right\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\gamma+2n})} + \langle f, Mh\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\gamma+2n})} \\
&= \left\langle \sum_{k=1}^{n} \mathcal{R}_k N^k f + f, \sum_{k=1}^{n} \mathcal{S}_k N^k Mh + Mh \right\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\gamma+2n})} \\
&= \langle \mathcal{A}f, M\mathcal{B}h\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\gamma+2n})}.
\end{aligned}$$

□

where M is a positive constant depend on α, γ. Let $g = Mh$ then we also have $g \in H(\mathbb{B}^d,\alpha) \subset H(\mathbb{B}^d,\beta)$ if $\beta > \alpha$. Therefore there exists $g \in H(\mathbb{B}^d,\beta)$ such that $F(f) = \langle \mathcal{A}f, \mathcal{B}g\rangle_{HL^2(\mathbb{B}^d,\mu_{\gamma+2n})} = \langle f, g\rangle_\gamma$, for all $f \in H(\mathbb{B}^d,\alpha)$.

The condition $\beta > \alpha$ restricts us to say that this theorem is valid only for $\beta > \alpha > 0$. However for $\alpha > \beta$ from above we get $H(\mathbb{B}^d, \beta)^* \subseteq H(\mathbb{B}^d, \alpha)$ and since H is reflexive Banach spaces therefore $H(\mathbb{B}^d, \alpha)^* \subseteq H(\mathbb{B}^d, \beta)^{**} = H(\mathbb{B}^d, \beta)$ which make the theorem to be valid for all $\alpha, \beta > 0$.

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CHAPTER 1

Introduction

Let $\mathbb{B}^d = \left\{ z = (z_1,\, z_2,\, \dots\, z_d) \in \mathbb{C}^d \colon \|z\| = \sqrt[d]{\sum_{i=1}^{d} |z_i|^2} < 1 \right\}$ be the open unit ball in $\mathbb{C}^n$. We define the measure

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For $\lambda > d$ and by the Riesz representation, any function $f \in \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ can be represented as

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where N denote the number operator

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Then $\langle f, g\rangle_\lambda = \langle Af, Bg\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\lambda+2n})}$ where

$$\begin{aligned} A &= \left(I + \frac{N}{\lambda+n}\right)\left(I + \frac{N}{\lambda+n+1}\right)\cdots\left(I + \frac{N}{\lambda+2n-1}\right) \\ B &= \left(I + \frac{N}{\lambda}\right)\left(I + \frac{N}{\lambda+1}\right)\cdots\left(I + \frac{N}{\lambda+n-1}\right) \end{aligned}$$

defines an inner product on $H(\mathbb{B}^d, \lambda)$ and , with respect to this inner product, $H(\mathbb{B}^d, \lambda)$ is a complete space whose reproducing kernel is also given by $K_\lambda(z, w) = \dfrac{1}{(1 - z \cdot \overline{w})^\lambda}$. Moreover, $H(\mathbb{B}^d, \lambda)$ is identical to $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ when $\lambda > d$.

By the definition of a generalized Bergman space. In this research, we will show that the duality of a generalized Bergman space can be proved by direct computation and boundedness of coefficients.

CHAPTER 2

Preliminaries

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(ii) $\|cx\| = |c|\|x\|$ for any $x \in X$ and $c \in \mathbb{F}$

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in X$.

A vector space equipped with a norm is called a **normed linear space**, or simply a **normed space**. Property (iii) is referred to as the *triangle inequality*.

Definition 2. The metric space (X, d) is said to be **complete** if every Cauchy sequence in X converges (that is has a limit which is an element of X). That is if $d(x_n, x_m) \to 0$ as $m, n \to \infty$ then $\{x_n\}$ must converge also in X.

Definition 3. A **Banach Space** is a normed linear space which is complete in the metric defined by its norm. That is $d(x, y) = \|x - y\|$.

Definition 4. An **inner product** on a vector space V is a function that associates a complex number $\langle u, v\rangle$ with each pair of vector u and v in V in such a way that the following axioms are satisfied for all vectors u, v and w in V and all scalars k.

(i) $\langle u, v\rangle = \langle v, u\rangle$

(ii) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

(iii) $\langle ku, v\rangle = k\langle u, v\rangle$

(iv) $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ if and only if $v = 0$.

A vector space equipped with an inner product is called an **inner product space**. So if we define $\|v\| = \sqrt{\langle v, v\rangle}$ then $\|\cdot\|$ is a norm on V.

Definition 5. For $1 \geq p < \infty$,the $\mathcal{L}^{\mathcal{P}}(X,\mu)$-space is the collection of all functions $f : X \to \mathbb{C}$ such that

$$\int_X \|f(z)\|^p d\mu(z) < \infty.$$

We define $L^p(X,\mu)$ to be the space of all equivalence classes of functions in $L^p(X,\mu)$ under the relation fg if and only if $f = g$ almost everywhere with respect to the measure μ

Definition 6. A **Hilbert space** is an inner product space which is complete with respect to the norm given by the inner product.

Theorem 1. (Riesz Representation) *If L is a bounded linear functional on a Hilbert space H, then there exists a unique $y \in H$ such that*

$$L(x) = \langle x, y\rangle \qquad \textit{for each } x \in H$$

Moreover $\|L\| = \|y\|$.

Theorem 2. (Hölder inequality) *If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq (\sum_{i=1}^{n} |a_i|^p)^{\frac{1}{p}} (\sum_{i=1}^{n} |b_i|^q)^{\frac{1}{q}}$$

Definition 7. Let X be a norm linear space. Denote by X^* the set of all bounded linear functional on X. We call X^* the **dual space** of X

Theorem 3. (Duality of Bergman spaces) *A Bergman space can be represented by the dual of another Bergman space by the following theorem. (See Zhu,K Theorem* 2.12*) For $\alpha, \beta > d$,*

$$\mathcal{H}L^2(\mathbb{B}^d, \mu_\alpha)^* = \mathcal{H}L^2(\mathbb{B}^d, \mu_\beta)$$

under the inner product

$$\langle f, g\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_\gamma)} = \int_{\mathbb{B}^d} f(z)\overline{g(z)}\, d\mu_\gamma(z),$$

for $f \in \mathcal{H}L^2(\mathbb{B}^d, \mu_\alpha)$, $g \in \mathcal{H}L^2(\mathbb{B}^d, \mu_\beta)$ *and* $\gamma = \dfrac{\alpha+\beta}{2}$.

Duality of generalized Bergman spaces. It should be noted that a Bergman space $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ is a closed subspace of the space $L^2(\mathbb{B}^d, \mu_\lambda)$. However, by its definition, $H(\mathbb{B}^d, \lambda)$ is not defined as a subspace of any L^2 space. Therefore the proof of the duality of Bergman spaces cannot be adopted to $H(\mathbb{B}^d, \lambda)$. However, the duality of generalized Bergman space can be proved by direct computation and boundedness of coefficients.

CHAPTER 3

Main Results

Theorem 4. *For* $\alpha, \beta > 0$

$$H(\mathbb{B}^d, \alpha)^* = H(\mathbb{B}^d, \beta)$$

under the inner product

$$\langle f, g\rangle_\gamma = \int_{\mathbb{B}^d} Af(z)\overline{Bg(z)}\, d\mu_{\gamma+2n}(z),$$

for $f \in H(\mathbb{B}^d, \alpha)$, $g \in H(\mathbb{B}^d, \beta)$ *and* $\gamma = \dfrac{\alpha+\beta}{2}$.

Proof. For each $g \in H(\mathbb{B}^d, \beta)$, we define $T_g \colon H(\mathbb{B}^d, \alpha) \to \mathbb{C}$ by

$$T_g(f) = \langle f, g\rangle_\gamma.$$

Next, we will prove that $T_g \in H(\mathbb{B}^d, \alpha)^*$. Consider

$$\begin{aligned}
|T_g(f)| &= |\langle f, g\rangle_\gamma| \\
&= |\langle Af, Bg\rangle_{\mathcal{H}L^2(\mathbb{B}^d, \mu_{\gamma+2n})}| \\
&= c_{\gamma+2n}\left|\int_{\mathbb{B}^d} Af(z)\overline{Bg(z)}(1-|z|^2)^{\gamma+2n}\,(1-|z|^2)^{-(d+1)}dz\right| \\
&\le c_{\gamma+2n}\int_{\mathbb{B}^d}(1-|z|^2)^{\frac{\alpha+2n}{2}}|Af(z)|(1-|z|^2)^{\frac{\beta+2n}{2}}\overline{|Bg(z)|}\,(1-|z|^2)^{-(d+1)}dz.
\end{aligned}$$

By Hölder's inequality,

$$
\begin{aligned}
|T_g(f)| &\leq c_{\gamma+2n}\left(\int_{\mathbb{B}^d}((1-|z|^2)^{\frac{\alpha+2n}{2}}|Af(z)|)^2\,(1-|z|^2)^{-(d+1)}dz\right)^{\frac{1}{2}} \\
&\quad\cdot\left(\int_{\mathbb{B}^d}((1-|z|^2)^{\frac{\beta+2n}{2}}\overline{|Bg(z)|})^2\,(1-|z|^2)^{-(d+1)}dz\right)^{\frac{1}{2}} \\
&= c_{\gamma+2n}\left(\int_{\mathbb{B}^d}|Af(z)|^2(1-|z|^2)^{\alpha+2n}(1-|z|^2)^{-(d+1)}dz\right)^{\frac{1}{2}} \\
&\quad\cdot\left(\int_{\mathbb{B}^d}\overline{|Bg(z)|}^2(1-|z|^2)^{\beta+2n}(1-|z|^2)^{-(d+1)}dz\right)^{\frac{1}{2}} \\
&= c_{\gamma+2n}\|Af(z)\|_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\alpha+2n})}\|Bg(z)\|_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\beta+2n})} \\
&= c_{\gamma+2n}\langle Af(z),Af(z)\rangle_{\alpha+2n}\langle Bg(z),Bg(z)\rangle_{\beta+2n}.
\end{aligned}
$$

By considering the coefficients in the operators A and B, there exist constants $C_A(n,\alpha)$ and $C_B(n,\beta)$ such that $\langle Af(z),Af(z)\rangle_{\alpha+2n}\leq C_A(n,\alpha)\langle f(z),f(z)\rangle_{\alpha+2n}$ and $\langle Bg(z),Bg(z)\rangle_{\beta+2n}\leq C_B(n,\alpha)\langle g(z),g(z)\rangle_{\beta+2n}$.
Therefore, $|T_g(f)|\leq C\|g\|_{\beta+2n}\|f\|_{\alpha+2n}$ where the constant C is independent of f.

Conversely, let $F\in H(\mathbb{B}^d,\alpha)^*$. By Riesz representation, there exists a function $h\in H(\mathbb{B}^d,\alpha)$ such that $F(f)=\langle f,h\rangle_\alpha$ for all $f\in H(\mathbb{B}^d,\alpha)$. To prove $H(\mathbb{B}^d,\alpha)^*=H(\mathbb{B}^d,\beta)$, we need a function $g\in H(\mathbb{B}^d,\beta)$, instead of $h\in H(\mathbb{B}^d,\alpha)$, such that $F(f)=\langle f,g\rangle_\gamma$. However, by manipulating the coefficients, we obtain that function g.

Consider, for $f\in H(\mathbb{B}^d,\alpha)$,

$$F(f)=\langle f,h\rangle_\alpha=\langle Af,Bh\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\alpha+2n})}.$$

Now the operator A and B can be distributed as

$$A=\sum_{k=1}^{n}R_kN^k+I \text{ and } B=\sum_{k=1}^{n}S_kN^k+I.$$

Therefore,

$$\begin{aligned}
F(f) &= \langle Af, Bh\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\alpha+2n})} \\
&= \left\langle \sum_{k=1}^{n} R_k N^k f + f, \sum_{k=1}^{n} S_k N^k h + h \right\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\alpha+2n})} \\
&= \left\langle \sum_{k=1}^{n} R_k N^k f, \sum_{k=1}^{n} S_k N^k h \right\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\alpha+2n})} + \left\langle \sum_{k=1}^{n} R_k N^k f, h \right\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\alpha+2n})} \\
&\quad + \left\langle f, \sum_{k=1}^{n} S_k N^k h \right\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\alpha+2n})} + \langle f, h\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\alpha+2n})} \\
&= \left\langle \sum_{k=1}^{n} \mathcal{R}_k N^k f, \sum_{k=1}^{n} \mathcal{S}_k N^k Mh \right\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\gamma+2n})} + \left\langle \sum_{k=1}^{n} \mathcal{R}_k N^k f, Mh \right\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\gamma+2n})} \\
&\quad + \left\langle f, \sum_{k=1}^{n} \mathcal{S}_k N^k Mh \right\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\gamma+2n})} + \langle f, Mh\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\gamma+2n})} \\
&= \left\langle \sum_{k=1}^{n} \mathcal{R}_k N^k f + f, \sum_{k=1}^{n} \mathcal{S}_k N^k Mh + Mh \right\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\gamma+2n})} \\
&= \langle \mathcal{A}f, M\mathcal{B}h\rangle_{\mathcal{H}L^2(\mathbb{B}^d,\mu_{\gamma+2n})}.
\end{aligned}$$

□

where M is a positive constant depend on α, γ. Let $g = Mh$ then we also have $g \in H(\mathbb{B}^d, \alpha) \subset H(\mathbb{B}^d, \beta)$ if $\beta > \alpha$. Therefore there exists $g \in H(\mathbb{B}^d, \beta)$ such that $F(f) = \langle \mathcal{A}f, \mathcal{B}g\rangle_{HL^2(\mathbb{B}^d,\mu_{\gamma+2n})} = \langle f, g\rangle_\gamma$, for all $f \in H(\mathbb{B}^d, \alpha)$.

The condition $\beta > \alpha$ restricts us to say that this theorem is valid only for $\beta > \alpha > 0$. However for $\alpha > \beta$ from above we get $H(\mathbb{B}^d, \beta)^* \subseteq H(\mathbb{B}^d, \alpha)$ and since H is reflexive Banach spaces therefore $H(\mathbb{B}^d, \alpha)^* \subseteq H(\mathbb{B}^d, \beta)^{**} = H(\mathbb{B}^d, \beta)$ which make the theorem to be valid for all $\alpha, \beta > 0$.

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